IDENTIFIABILITY OF DISTRIBUTED PARAMETERS FOR A CLASS OF QUASILINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. Considering differential equations of second order which contain a term in the form $(au_x)_x + bu$, where $a = a(x,u)$ and $b = b(x,u)$, conditions for the identifiability of the coefficients a and b are given.

1. INTRODUCTION

Modelling of one-dimensional diffusive-like systems leads to a parabolic differential equation, which for a linear isotropic medium (for example a confined aquifer), can be written in the form

$$(a(x,u)u_x(x,t))_x + f(x,t) = u_t(x,t).$$

The coefficient depends upon the space variable x and on the potential $u(x,t)$. Our aim is to determine the physical parameter a from measurements of u (potential) and f (source term). In a lot of practical problems, the knowledge of such a parameter is of interest. Examples include the heat conduction in solids, fluid flow through porous media, groundwater or oil reservoir problems and pollutant diffusion in absence of convection [2],[1]. From such considerations arises the well known inverse problem of parameter identification. Such a problem is ill-posed, since a solution, if it exists, need not be unique and in general does not depend continuously on the data. Identifiability is equivalent to the uniqueness of the solution of the inverse problem in its direct formulation.

From the mathematical point of view we can investigate inverse problems for a wider class of equations as above by a unified approach. Let us formulate this class of problems. We consider the quasilinear boundary value problem (steady- state 1D case)

$$(a(x,u(x))u_x(x))_x + b(x,u(x))u(x) + f(x) = 0 \qquad x \in D_0 = (d_1,d_2) \tag{1.1}$$

$$u(d_1) = g_1 \qquad u(d_2) = g_2 \qquad g_1 \neq g_2, \tag{1.2}$$

the quasilinear parabolic initial boundary value problem

$$(a(x,u(x,t))u_x(x,t))_x + b(x,u(x,t))u(x,t) + f(x,t) = u_t(x,t) \tag{1.3}$$

$$(x,t) \in D_T = (d_1,d_2) \times (0,T)$$

$$u(d_1,t) = g_1(t) \qquad u(d_2,t) = g_2(t) \qquad t \in [0,T] \tag{1.4}$$

$$u(x,0) = \varphi_1(x) \qquad x \in [d_1,d_2] \tag{1.5}$$

and the quasilinear hyperbolic initial boundary value problem

$$(a(x,u(x,t))u_x(x,t))_x + b(x,u(x,t))u(x,t) + f(x,t) = u_{tt}(x,t) \qquad (x,t) \in D_T \tag{1.6}$$

with boundary conditions (1.4) and the initial conditions

$$u(x,0) = \varphi_1(x) \qquad u_t(x,0) = \varphi_2(x) \qquad x \in [d_1,d_2]. \tag{1.7}$$

The following assumptions and assertions we formulate for all three problems in common, where, in the steady-state 1D case, the variable t does not appear.

For the direct problem we consider: given $a, b, f, g_1, g_2, \varphi_1, \varphi_2,$ find u. We define the real numbers

$$v_1 = \min_{(x,t)\in\overline{D_T}} u(x,t), \qquad v_2 = \max_{(x,t)\in\overline{D_T}} u(x,t),$$

and suppose that:

(i) $v_1 < v_2$
(ii) $a(x,u) > 0 \qquad \forall (x,u) \in [d_1,d_2]\times[v_1,v_2]$
(iii) $g_1(0) = \varphi_1(d_1), \qquad g_2(0) = \varphi_1(d_2)$
(iv) $a, b, f, g_1, g_2, \varphi_1, \varphi_2$ are sufficiently smooth functions of their arguments, respectively such that each of the direct problems (1.1), (1.2), (1.3), (1.4), (1.5) or (1.6), (1.4), (1.7) has a unique classical solution u.

From the continuity of u, it follows that $range\{u\} = [v_1, v_2]$. The assumption (iii) does not appear for the problem (1.1), (1.2).

Now we formulate two inverse problems:

(I) Given $b, f, g_1, g_2, \varphi_1, \varphi_2, u,$ find a.
(II) Given $a, f, g_1, g_2, \varphi_1, \varphi_2, u,$ find b.

There exist simple examples that, even if u is completely known it does not follow the uniqueness of one of the coefficients a or b. For that reason, we introduce the concept of identifiability [9].

Definition 1. *Let $u_j(x,t)$ be the solution of the direct problem which corresponds to $a_j = a_j(x,u)$ $(j = 1,2)$. We call the coefficient a identifiable if from $u_1(x,t) = u_2(x,t)$ for every $(x,t) \in \overline{D_T}$ it follows that $a_1(x,u) = a_2(x,u)$ for all $(x,u) \in [d_1,d_2]\times[v_1,v_2]$.*

Analogously, the identifiability of the coefficient b can be defined. We have two important special cases, namely

1. $a = a(x) \qquad b = b(x)$, which corresponds to a linear physical system, and
2. $a = a(u) \qquad b = b(u)$.

First results about identifiability have been obtained by Kitamura and Nakagiri [8] for space dependent coefficients in an one-dimensional parabolic equation. A generalization of this results in connection with applications is given by Guidici [6]. For a more dimensional elliptic equation, such results can be found by Chicone and Gerlach [3], [4]. The identifiability of the coefficient $a = a(u)$ in a one-dimensional parabolic equation was considered also (see [5]). Identifiability results for coefficients $a = a(x), b = b(x)$ or $a = a(u), b = b(u)$ in the formulated above class of problems by a unified approach has been received by Handrock-Meyer [7].

The present paper generalizes these results on coefficients of the form

$$a = a(x,u) \qquad b = b(x,u).$$

In section 2, some lemmas are proven which are needed in later sections. In section 3, several results about the identifiability of the coefficient a or b are presented. The simultaneous identifiability of a and b is not investigated in this paper.

2. Some lemmas

For equations with the left hand side having the form (1.1), (1.3) or (1.6), the following lemma is true.

Lemma 1. *$u_1(x,t) = u_2(x,t)$ for all $(x,t) \in \overline{D_T}$ holds if and only if*

$$(a_{12}(x,u_1(x,t))(u_1(x,t))_x)_x + b_{12}(x,u_1(x,t))u_1(x,t) = 0 \qquad \forall (x,t) \in D_T, \tag{2.1}$$

where $a_{12}(x,u) = a_1(x,u) - a_2(x,u)$ and $b_{12}(x,u) = b_1(x,u) - b_2(x,u)$.

Proof.

1. Let $u_1(x,t) = u_2(x,t)$ for all $(x,t) \in \overline{D_T}$. From (1.1)(1.3) and (1.6) it follows that

$$(a_i(x,u_i)(u_i)_x)_x + b_i(x,u_i)u_i + f = \begin{cases} 0 \\ (u_i)_t \\ (u_i)_{tt} \end{cases}$$

$(i = 1,2)$ and by subtraction

$$(a_1(x,u_1)(u_1)_x)_x - (a_2(x,u_2)(u_2)_x)_x + b_1(x,u_1)u_1 - b_2(x,u_2)u_2 = \begin{cases} 0 \\ (u_1)_t - (u_2)_t \\ (u_1)_{tt} - (u_2)_{tt} \end{cases} \tag{2.2}$$

Setting $u_1 = u_2$ in (2.2), we obtain (2.1).

2. Let (2.1) hold. We prove the assertion for the parabolic case (1.3)–(1.5). For the problems (1.1), (1.2) and (1.6), (1.4), (1.7), the statement can be shown in the same way. The function $u_2(x,t)$ is a solution of the initial boundary value problem

$$(a_2(x,u_2)(u_2)_x)_x + b_2(x,u_2)u_2 + f = (u_2)_t \qquad (x,t) \in D_T$$

$$u_2(0,t) = g_1(t) \qquad u_2(1,t) = g_2(t) \qquad t \in [0,T]$$

$$u_2(x,0) = \varphi_1(x) \qquad x \in [d_1,d_2].$$

The function $u_1(x,t)$ satisfies the same initial and boundary conditions. From (2.1) we obtain

$$(a_2(x,u_1)(u_1)_x)_x + b_2(x,u_1)u_1 + f = (a_1(x,u_1)(u_1)_x)_x + b_1(x,u_1)u_1 + f = (u_1)_t.$$

Hence $u_1(x,t)$ is a solution of the above initial boundary value problem and $u_1(x,t) = u_2(x,t)$ for all $(x,t) \in \overline{D_T}$.

For $b(x,u(x,t)) \equiv 0$ we formulate Lemma 1 in a more convenient form

Lemma 2. *$u_1(x,t) = u_2(x,t)$ for all $(x,t) \in \overline{D_T}$ holds if and only if*

$$a_{12}(x,u_1(x,t))(u_1(x,t))_x = a_{12}(x_0,u_1(x_0,t))(u_1(x_0,t))_x \tag{2.3}$$

for all $x, x_0 \in [d_1,d_2]$ and every fixed $t \in (0,T)$.

Proof.

1. Integration of (2.1) yields

$$a_{12}(x, u_1(x,t))(u_1(x,t))_x = C(t).$$

Setting $x = x_0$ for every fixed t, we obtain (2.3).

2. Differentiation with respect to x leads to (2.1).

For the investigation of the problem (I) it is useful to introduce the following sets

$$P(t) = \{x \in [d_1, d_2] \mid u_x(x,t) = 0\} \tag{2.4}$$

$$Q(t) = [d_1, d_2] - P(t)$$

$$V(x) = \{t \in [0,T] | u_x(x,t) = 0\}$$

$$W(x) = [0,T] - V(x)$$

$$N_u = \{s \in [v_1, v_2] \mid \exists (x,t) \in \overline{D_T} : s = u(x,t) \wedge u_x(x,t) \neq 0\}.$$

Lemma 3. *If there exists a point $t_1 > 0$ such that $\overline{Q(t_1)} = [d_1, d_2]$ and for every $x \in [d_1, d_2]$ it holds that $\overline{W(x)} = [0,T]$, then $\overline{N_u} = [v_1, v_2]$.*

Proof. Let there be given any $\epsilon > 0$ and any $z_0 \in [v_1, v_2]$. Because the $\underline{range\{u\}} = [v_1, v_2]$ there exists $(x_0, t_0) \in \overline{D_T}$ such that $z_0 = u(x_0, t_0)$. From the assumption $\overline{Q(t_1)} = [d_1, d_2]$ we obtain: For every $\delta > 0$ and every $x \in [d_1, d_2]$, especially for $x_0 \in [d_1, d_2]$, there exists a point $y \in Q(t_1)$ such that $\mid x_0 - y \mid < \delta$, where $u_x(y, t_1) \neq 0$. Furthermore, the assumption $\overline{W(x)} = [0,T]$ for every $x \in [d_1, d_2]$ gives $\overline{W(y)} = [0,T]$, i.e. for every $\delta > 0$ and every $t \in [0,T]$, especially for $t_0 \in [0.T]$, there exists a point $\tau \in W(y)$ such that $\mid t_0 - \tau \mid < \delta$, where $u_x(y, \tau) \neq 0$. Setting $s = u(y, \tau)$ we obtain $s \in N_u$. Moreover, putting $P_0 = (x_0, t_0), P = (y, \tau)$ we have

$$\mid P_0 - P \mid = \sqrt{(x_0 - y)^2 + (t_0 - \tau)^2} < \sqrt{2}\delta$$

and from the continuity of the function u it follows that

$$\mid z_0 - s \mid = \mid u(x_0, t_0) - u(y, \tau) \mid < \epsilon.$$

This proves the lemma.

Analogously we can show

Lemma 4. *If there exists a point $x_1 > 0$ such that $\overline{W(x_1)} = [0,T]$ and for every $t \in [0,T]$ it holds that $\overline{Q(t)} = [d_1, d_2]$, then $\overline{N_u} = [v_1, v_2]$.*

Lemma 5. *If $P(t) = \phi$ for all $t \in [0,T]$, then $\cup Q(t) = [d_1, d_2]$ and $N_u = [v_1, v_2]$.*

Proof. From $P(t) = \phi$ for all $t \in [0,T]$ it follows that

$$\cap P(t) = \cap([d_1, d_2] - Q(t)) = [d_1, d_2] - \cup Q(t) = \phi.$$

Moreover, from $P(t) = \phi$ for all $t \in [0,t]$, we have $u_x(x,t) \neq 0$ for all $(x,t) \in \overline{D_T}$ and $N_u = [v_1, v_2]$.

Remark 1. *If $P(t) \neq \phi$ for all $t \in [0,T]$ then in the dependence of the structure of the set $\cap P(t)$ the density of N_u in $[v_1, v_2]$ follows or not. We consider two cases:*

case 1: Let $P(t) = \{x_1\}$ for all $t \in [0,T]$, then $\cup P(t) = \{x_1\}$ and $u_x(x,t) = 0$ only on the set $\{x_1\} \times [0,T]$. Now we can choose for any $\epsilon > 0$ and any $z_0 \in [v_1, v_2]$ with $z_0 = u(x_0, t_0)$, a point $s = u(x,t)$ with $s \in N_u$ such that

$$\mid z_0 - s \mid < \epsilon.$$

Hence $\overline{N_u} = [v_1, v_2]$.

case 2: Let $\cap P(t) = [\alpha, \beta]$. Then we have $[\alpha, \beta] = [d_1, d_2] - \cup Q(t)$ and $\cup Q(t)$ is not dense in $[d_1, d_2]$. We show that N_u is not dense in $[v_1, v_2]$ too. For $x_0 \in [\alpha, \beta]$ we choose $\epsilon > 0$ in such a way that $(x_0 - \epsilon, x_0 + \epsilon) \subset [\alpha, \beta]$. By (2.4) we can verify that, for every $(x,t) \in (x_0 - \epsilon, x_0 + \epsilon) \times (0,T)$, $\quad u_x(x,t) = 0$ holds and, if we set $s = u(x,t)$, then $s \in [v_1, v_2] - N_u$ for all $(x,t) \in (x_0 - \epsilon, x_0 + \epsilon) \times (0,T)$ Thus for the chosen ϵ and s, the inequality $\mid s - z \mid \geq \epsilon$ is valid for any $z \in N_u$.

3. Identifiability results

3.1 Identifiability of $a(x,u)$

Throughout this subsection, let

$$u_1(x,t) = u_2(x,t) = u(x,t) \quad \forall (x,t) \in \overline{D_T}$$

with $b(x,u)$ is known or $b(x,u) = 0$ for all $(x,u) \in [d_1, d_2] \times [v_1, v_2]$. Then from lemma 1 it follows that

$$(a_{12}(x, u(x,t)) u_x(x,t))_x = 0 \quad \forall (x,t) \in D_T. \tag{3.1}$$

If we can deduce from (3.1) that $a_{12}(x,u) = 0$ for all $(x,u) \in [d_1, d_2] \times [v_1, v_2]$, then $a(x,u)$ is identifiable. The identifiability of $a(x,u)$ depends essentially on the structure of the zeros of the derivative $u_x(x,t)$ and of the density of the sets $\cup Q(t)$ in $[d_1, d_2]$ as well as N_u in $[v_1, v_2]$, respectively.

Theorem 1. *We suppose*

$$P(t) \neq \phi \quad \forall t \in [0,T] \tag{3.2}$$

$$\overline{\cup Q(t)} = [d_1, d_2] \tag{3.3}$$

$$\overline{N_u} = [v_1, v_2]. \tag{3.4}$$

Then $a(x,u)$ is identifiable.

Proof. From (3.1) it follows that for every $t \in [0,T]$ there exists a point $x_t \in [d_1, d_2]$ such that $u_x(x_t, t) = 0$. We set $x_0 = x_t$ in (2.3) and obtain

$$a_{12}(x, u(x,t)) u_x(x,t) = 0 \quad \forall (x,t) \in D_T.$$

Set $K = \cup Q(t)$. For any $x \in K$ there exists some $t(x) > 0$ such that $x \in Q(t)$, i.e. $u_x(x,t) \neq 0$ and $u = u(x,t) \in N_u$ for all such (x,t). Thus, $a_{12}(x,u) = 0$ for all $(x,u) \in K \times N_u$. Now from the assumptions (3.3), (3.4) and from the continuity of $a(x,u)$ as a function of two variables, it follows that

$$a_{12}(x,u) = 0 \quad \forall (x,u) \in [d_1, d_2] \times [v_1, v_2].$$

The assumption (3.2) can be replaced by an additional information on the coefficient $a(x,u)$. Unfortunately, in practice, knowledge about the physical parameters is generally not available. For the completeness of the mathematical results, we provide such a theorem.

Theorem 2. *Suppose that for every $t \in [0,T]$ there exists a point $x_t \in [d_1, d_2]$ such that*

$$a_1(x_t, u(x_t,t)) = a_2(x_t, u(x_t,t)). \tag{3.5}$$

Moreover, let one of the following conditions be satisfied:

(i) *The assumptions of the lemma 3.*
(ii) *The assumptions of the lemma 4.*
(iii) $P(t) = \phi \quad \forall t \in [0,T]$.

Then

$$a_1(x,u) = a_2(x,u) \quad \forall (x,u) \in [d_1,d_2] \times [v_1,v_2].$$

Proof. From (3.5) we obtain, as in theorem 1,

$$a_{12}(x, u(x,t)) u_x(x,t) = 0 \quad \forall (x,t) \in D_T.$$

For every $u \in N_u$, there exists $(x,t) \in D_T$ such that $u = u(x,t)$ with $u_x(x,t) \neq 0$ and

$$a_{12}(x, u(x,t)) = 0 \quad \text{for all these} \quad (x,t). \tag{3.6}$$

Let (i) be satisfied. From (3.6) it follows especially that

$$a_{12}(x, u(x,t_1)) = 0 \quad \forall x \in Q(t_1).$$

Now lemma 3 and the continuity of $a(x,u)$ give

$$a_{12}(x,u) = 0 \quad \forall (x,u) \in [d_1,d_2] \times [v_1,v_2]$$

as in the proof of the theorem 1.

If (ii) is satisfied, then from (3.6) we can deduce

$$a_{12}(x_1, u(x_1,t)) = 0 \quad \forall t \in W(x_1)$$

and the assertion from lemma 4 follows as above.

Finally, let (iii) be satisfied. In this case,

$$a_{12}(x, u(x,t)) = 0 \quad \forall (x,t) \in D_T.$$

holds. Hence we obtain the desired result from lemma 5 and from the continuity of $a(x,u)$.

Remark 2. *The assumptions (3.3) and (3.4) in theorem 1 can be replaced by one of the conditions (i) or (ii) in theorem 2 and vice versa.*

Now let us consider some cases of non-identifiability. If the condition (iii) in theorem 2 is fulfilled, then (3.5) must not be omitted.

Theorem 3. *Let the first derivative with respect to x of the function $u_1(x,t)$ have a representation of the form*

$$(u_1(x,t))_x = f(u) g(x) h(t),$$

where $f(u) > 0$ *for all* $u \in [v_1, v_2]$, $g(x) > 0$ *for all* $x \in [d_1, d_2]$, $f \in C^1([v_1, v_2])$, $g \in C^1([d_1, d_2])$. *Then there exists an* a_2 *such that* $u_1(x,t) = u_2(x,t)$ *for all* $(x,t) \in \overline{D_T}$ *and* $a_1(x,u) \neq a_2(x,u)$ *for all* $(x,u) \in [d_1, d_2] \times [v_1, v_2]$.

Proof. We set

$$a_2(x,u) = a_1(x,u) + \frac{1}{f(u)g(x)}.$$

Then $a_2(x,u) > 0$ and $a_1(x,u) \neq a_2(x,u)$ for all $(x,u) \in [d_1, d_2] \times [v_1, v_2]$. Furthermore, we obtain

$$(a_{12}(x,u_1)(u_1(x,t))_x)_x = 0 \quad \forall (x,t) \in D_T.$$

Then lemma 1 implies $u_1(x,t) = u_2(x,t)$ for all $(x,t) \in \overline{D_T}$.

For the illustration of the theorems 2 and 3, we give an

Example: We consider the parabolic problem (1.3)–(1.5), where $b(x,u) \equiv 0$, $d_1 = 0$, $d_2 = 1$, $T = 1$, $f(x,t) \equiv 0$, $g_1(t) = (t+1)$, $g_2(t) = 1/4(t+1)$, $\varphi(x) = (1+x)^{-2}$. Let

$$u(x,t) = \frac{t+1}{(x+1)^2} \tag{3.7}$$

be a solution of the equation (1.3) which fulfilles the boundary value conditions (1.4) and the initial condition (1.5), respectively. From the maximum principle for quasilinear parabolic equations [10], we obtain $v_1 = 1/4$, $v_2 = 2$. Furthermore, we have

$$u_x(x,t) = -\frac{2(t+1)}{(x+1)^3} = u^{3/2}(-2(t+1)^{-1/2}) \neq 0 \quad \forall (x,t) \in (0,1) \times (0,1)$$

and the assumptions of theorem 3 are satisfied. We determine $a(x,u)$ as a solution of the first order partial differential equation

$$(a(x,u)u_x)_x = u_t$$

which can be written also in the form

$$u_x a_x + u_x^2 a_u + u_{xx} a = u_t \tag{3.8}$$

with $u(x,t)$ in the form (3.7). From the special form of the coefficients in equation (3.8), we can easily see that we obtain only one first integral

$$\frac{a}{(x+1)^3} - \frac{1}{2u(x+1)^3} = C.$$

For this reason it is sufficient to give an initial condition for equation (3.8) in the following form: We suppose that there exists a point x_0 such that for every $t \in [0,T]$ the value $a(x_0, u(x_0,t)) = a_0(t)$ is known (in this case the assumption (3.5) in theorem 2 is obviously fulfilled). In particular, choosing $x_0 = 1$, i.e., $a(1, u(1,t)) = (t+1)^{-1}$, we obtain

$$a(x,u) = \frac{1}{2u}(1 - \frac{(x+1)}{4}).$$

Also, the assumption (3.4) in theorem 1 must not be omitted.

Theorem 4. *If* $N_{u_1} = \{s \in [v_1, v_2] | \exists (x,t) \in \overline{D_T} : s = u_1(x,t) \wedge (u_1(x,t))_x \neq 0\}$ *is not dense in* $[v_1, v_2]$. *Then* $a(x,u)$ *is not identifiable.*

Proof. Since N_{u_1} is not dense in $[v_1, v_2]$, we have an interval $Y = (x_0 - \epsilon, x_0 + \epsilon) \subset [v_1, v_2] - N_{u_1}$, i.e. for every $v \in Y$ there exists $(x,t) \in \overline{D_T}$ with the properties $v = u_1(x,t)$ and $(u_1(x,t))_x = 0$. We set $v_0 = u_1(x_0, t_0)$. Because $u_1(x,t)$ is a continuous function from $[d_1, d_2] \times [v_1, v_2]$ into $[v_1, v_2]$ the inverse image $u_1^{-1}(Y)$ of Y under u_1 is open in $[d_1, d_2] \times [v_1, v_2]$ [11]. Then there exists an open ball $S_\delta(\vec{r_0}) \subset u_1^{-1}(Y)$, where $\vec{r_0} = (x_0, t_0)$. We denote by $X = (x_0 - \delta, x_0 + \delta)$.

Let be $p(x,u) \in C^1([d_1, d_2] \times [v_1, v_2])$, where $supp\{p\} \subset X \times Y$, $p(x_0, v_0) > 0$ and $p(x,u) \geq 0 \quad \forall (x,u) \in [d_1, d_2] \times [v_1, v_2]$. We set

$$a_2 = a_1 + p.$$

If $(x,t) \in D_T$ such that $x \in X$ and $u_1(x,t) \in Y$, i.e. $(x, u_1) \in X \times Y$, then $(u_1(x,t))_x = 0$. If $(x,t) \in D_T$ such that $x \notin X$, then $(x, u_1) \notin X \times Y$ and $p(x, u_1(x,t)) = 0$. Now for all $(x,t) \in (d_1, d_2) \times (0, T)$

$$a_{12}(x, u_1(x,t))(u_1(x,t))_x = 0$$

holds and from lemma 1 it follows $u_1(x,t) = u_2(x,t)$ for all $(x,t) \in [d_1, d_2] \times [0, T]$, but,

$$a_1(x,u) = a_2(x,u) \quad \forall (x,u) \in [d_1, d_2] \times [v_1, v_2].$$

is not valid.

3.2 Identifiability of $b(x,u)$

We suppose again that

$$u_1(x,t) = u_2(x,t) = u(x,t) \quad \forall (x,t) \in \overline{D_T}$$

with $a(x,u)$ is known or $a(x,u) = 0$ for all $(x,u) \in [d_1, d_2] \times [v_1, v_2]$. From lemma 1 it follows

$$b_{12}(x,u)u(x,t) = 0 \quad \forall (x,t) \in D_T. \tag{3.9}$$

Moreover, we introduce the set

$$H(t) = \{x \in [d_1, d_2] | u(x,t) \neq 0\}.$$

Remark 3. $\overline{\cup H(t)} = [d_1, d_2]$ *holds if and only if* $[v_1, v_2] - \{0\} \neq 0$.

Indeed, if $\overline{\cup H(t)} = [d_1, d_2]$, then there exists $x_0 \in H(t)$ for some t_0 with $u(x_1, t_0) \neq 0$ whence it follows $[v_1, v_2] - \{0\} \neq \phi$.

Conversely, if we suppose $[v_1, v_2] - \{0\} = \phi$, then $u(x,t) = 0$ for every $(x,t) \in D_T$ and $\cup H(t) = \phi$.

Theorem 5. *$b(u,u)$ is identifiable if and only if* $\overline{\cup H(t)} = [d_1, d_2]$.

Proof.

1. We assume that $\overline{H(t)}$ is not dense in $[d_1, d_2]$. Then there exists an interval $X = (x_0 - \delta, x_0 + \delta) \subset [d_1, d_2] - \cup Q(t)$ with $u_1(x,t) = 0$ for all $(x,t) \in X \times [0, T]$. We set $q(x,u) \in C([d_1, d_2] \times [v_1, v_2]$,where $supp\{q\} \subset X \times \{0\}$, $q(x_0, 0) > 0$ $q(x,u) \geq 0$, $(x,u) \in [d_1, d_2] \times [v_1, v_2]$. We set

$$b_1 = b_2 + q.$$

If $(x,t) \in D_T$ such that $x \in X$, then $u_1(x,t) = 0$. If $(x,t) \in D_T$ such that $x \notin X$ then $(x, u_1) \notin X \times \{0\}$ and $q(x, u_1(x,t)) = 0$. Now for all $(x,t) \in (d_1, d_2) \times (0,T)$,

$$b_{12}(x, u_1(x,t)) u_1(x,t) = 0$$

and from lemma 1 it follows $u_1(x,t) = u_2(x,t)$ for all $(x,t) \in [d_1, d_2] \times [0,T]$, but

$$b_{12}(x,u) = 0 \quad \forall (x,u) \in [d_1, d_2] \times [v_1, v_2]$$

is not valid.

2. If $\overline{\cup H(t)} = [d_1, d_2]$, then for every $x \in H(t)$, $\quad u(x,t) \neq 0$ holds. Now it follows from (3.9) that

$$b_{12}(x,u) = 0 \quad \forall (x,u) \in \cup H(t) \times [v_1, v_2] - \{0\}.$$

From the density of $\cup H(t)$ in $[d_1, d_2]$ and the continuity of $b(x,u)$ we obtain

$$b_{12}(x,u) = 0 \quad \forall (x,u) \in [d_1, d_2] \times [v_1, v_2].$$

Remark 4. *Analogously as in theorem 5 we can obtain identifiability results for the coefficients a or b in the differential equations (1.1), (1.3), (1.6), which depends on n space variables* $x = (x_1, ... x_n)$ *if the left hand side is given in the non-divergent form*

$$a(x, u(x,t)) \Delta u(x,t) + b(x, u(x,t)) u(x,t) + f(x,t).$$

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